From last time, we have two questions. First, why is the dimension of a motive well defined? Second, how does it behave under $\otimes, \oplus$? We answer the 2nd first.

Direct Sum

Prop: Let M, N be motives. Then if M & N are evenly (oddly) f.d., then so is $M \oplus N$. If M, N are f.d., then $\dim M \oplus N \leq \dim M + \dim N$.

Proof: Suppose $\dim_+ M = m$, $\dim_- N = n$. Then we aim to show $\Lambda^{n+m+1}(M \oplus N) = 0$. By a direct computation shows:

$$\Lambda^{n+m+1}(M \oplus N) \cong \bigoplus_{\substack{r+s \\ = n+m+1}} \Lambda^r M \otimes \Lambda^s N = 0$$

by an index count. An identical argument proves the converse. ∎

Tensor Product

Def: For λ a partition of n, $M = (X, p, m)$, define $\pi_\lambda M = (X^n, d_\lambda(M) \circ p^n, nm)$.

Lemma (Vanishing): Let $q \geq n$ and λ a partition of q. Then

1) $\mathrm{Sym}^{n+1}(M) = 0$ + $\lambda_1 > n \Rightarrow \pi_\lambda M = 0$
2) $\Lambda^{n+1}(M) = 0$ + $\lambda_{n+1} \neq 0 \Rightarrow \pi_\lambda M = 0$.

Let T be the Young diagram according to λ. Define $R_\lambda(T) = \{\sigma \in S_n \mid \sigma \text{ permutes only elements in rows}\}$, $C_\lambda(T) = \{\sigma \in S_n \mid \sigma \text{ '' '' columns}\}$.

Now define $a_\lambda(T) = \sum_{\sigma \in R_\lambda(T)} \sigma$, $b_\lambda(T) = \sum_{\sigma \in C_\lambda(T)} \mathrm{sign}(\sigma)\, \sigma$, $c_\lambda(T) = a_\lambda(T)\, b_\lambda(T)$.
Recall from Rep. Theory:

1) $V \in \mathrm{Vect}_{\mathbb{Q}}$, and $S_n \circlearrowright V^{\otimes n}$ naturally. Then $\mathrm{Im}(a_\lambda(T)) = \mathrm{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_s}(V)$ and $\mathrm{Im}(b_\lambda(T)) = \Lambda^{\mu_1} V \otimes \cdots \otimes \Lambda^{\mu_n} V$.
2) $c_\lambda(T) \circ c_\lambda(T) = \eta_\lambda(T)\, c_\lambda(T)$, for $\eta_\lambda(T) \neq 0$.
3) $R_{S_n} c_\lambda(T)$ is an irred. R_{S_n}-module.
4) $R_{S_n} c_\lambda(T) \cong R_{S_n} c_\mu(T) \iff \lambda = \mu$.
5) e_λ is a linear combination of the $c_\lambda(T)$.
6) $e_{\underbrace{(1,\dots,1)}_{n^2}} \cdot (e_\lambda \otimes e_\mu) = \begin{cases} 0 & \text{if } \lambda \neq \mu^T \\ e_{(1,\dots,1)} & \text{else.} \end{cases}$

Now we have the following theorem:

Thm: Let M, N be f.d. motives. Then $\dim M \otimes N \leq \dim M \cdot \dim N$.

Proof: See Murre ∎.

Now we have the following goal: We had defined $\dim M = \dim_+ M_+ + \dim_- M_-$ for some decomposition $M = M_+ \oplus M_-$, yet we said nothing about this decomposition. We now aim to show it's well-defined.

Before we do this however, lets take a brief detour to setup some tools. Recall that if X is Sm. Proj. $/k$, and $Z \subset X$ is a cycle, then we say Z is smash nilpotent, $Z \sim_{\otimes} 0$ if $Z^n \sim_{\mathbb{Q}} 0$ on X^n for some $n \geq 1$.

Now in $Mot_\sim$, we have $Hom((X,p,m),(Y,q,n)) = q \circ Corr_\sim^{n-m}(X,Y) \circ p$, and the space of correspondences $Corr_\sim^r(X_d, Y) = C_\sim^{d+r}(X_d \times Y; \mathbb{Q})$, extended linearly. Since morphisms between motives are themselves cycles, we make the following definition:

Def: A morphism $f: M \to N$ in $Mot_\sim(k)$ is called smash-nilpotent if for some $n \geq 1$, the associated correspondence is smash-nilpotent w.r.t. $\sim$ (may not be $\sim_{\mathbb{Q}}$!).

This means that if Γ_f is the correspondence for f, $\Gamma_f \times \cdots \times \Gamma_f \sim 0$ in $C_\sim(X^n \times Y^n)_{\mathbb{Q}}$. This is exactly the same as $f^{\otimes n} = f \times \cdots \times f$ vanishing in the n^{th} tensor product of motives. We have the following obvious lemma:

Lemma: Let $f, g: M \to N$ be smash-nilpotent. Then so are $f+g$, $f-g$.

Proof: Let Γ_f, Γ_g be the associated correspondences. Then one checks that

$$(\Gamma_f + \Gamma_g)^n = \sum \binom{n}{r} \Gamma_f^r \times \Gamma_g^{n-r}$$

which can be made ~ 0 for sufficiently large n. The other is similar. ▨

While this result was simple, the real focus of this detour is the following:

Theorem: Let $f: M \to M$ be a smash-nilpotent morphism in $Mot_\sim(k)$. Then $f^{(n)} = f \circ \cdots \circ f = 0$. That is, smash-nilpotence $\Rightarrow$ nilpotence.

This in turn is implied by

Prop: Let $f: M \to N$ in $Mot_\sim(k)$ be smash nilpotent of order n, and let $g_i: N \to M$, $i = 1, \dots, n-1$ be morphisms. Then $f \circ g_{n-1} \circ f \cdots f \circ g_1 \circ f$ vanishes.

Clearly by taking $N = M$, $g_i = id$ we recover the theorem above, so now we prove the proposition.

Proof: To illustrate how this is proven, consider just $f \circ g_1 \circ f$. $f^{\otimes 2} = 0$ Denote their correspondences by Γ_f, Γ_{g_1}. Then by definition, if $M = (X, p, -)$ and $N = (Y, q, -)$, $f = q \circ \Gamma_f \circ p$ and $g_1 = p \circ \Gamma_{g_1} \circ q$. Hence $f \circ g_1 = q \circ \Gamma_f \circ p \circ \Gamma_{g_1} \circ q \in Corr(X, X)$. If we omit the projectors for a moment, and set π_{ijk} the projections from $X \times Y \times X \times Y$, s_{ij} the projections from $X \times Y \times X$, and p_{ij} projections from $X \times X \times Y$, then consider the cycles:

$$\alpha = \pi_{123}^*(s_{12}^* \Gamma_f \cdot s_{23}^* \Gamma) \; (= ((\Gamma_f \times X) \cdot (X \times \Gamma_g)) \times Y) \in C_\sim(X \times Y \times X \times Y)$$
$$\beta = p_{23}^* \Gamma_f \; (= X \times \Gamma_f) \in C_\sim(X \times X \times Y).$$

Now $\alpha \cdot \pi_{134}^*(\beta) = (\Gamma_f \times X \times Y) \cdot (X \times \Gamma_g \times Y) \cdot (X \times Y \times \Gamma_f) = (\Gamma_f \times \Gamma_f) \cdot (X \times \Gamma_g \times Y) = 0$ as $\Gamma_f \times \Gamma_f = 0$.

Now use the projection formula: $0 = (\pi_{134})_*(\alpha \cdot \pi_{134}^*(\beta)) = (\pi_{134})_* \alpha \cdot \beta$ and note that since $\pi_{134} = S_{13} \times id_4$, $(\pi_{134})_*(\alpha) = \pi_{123}^*(\Gamma_g \circ \Gamma_f) = (\Gamma_g \circ \Gamma_f) \times Y$. Then on $X \times X \times Y$, we have:

$$(\pi_{134})_* \alpha \cdot \beta = \{(\Gamma_g \circ \Gamma_f) \times Y\} \cdot (X \times \Gamma_f) = 0.$$

finally, apply $(S_{13})_*$ to the above $(\pi_{134} = S_{13} \times id_4)$ to get $\Gamma_f \circ \Gamma_g \circ \Gamma_f = 0$. ▨

Now let us turn back to the finite dimensionality of motives, we begin with a crucial result.

Prop: Let M, N be two f.d. motives of different parity (M evenly, N oddly for example), and $f: M \to N$. Then f is smash nilpotent, $f^{\otimes l} = 0$ if $l > \dim M \cdot \dim N$.

Proof: Set $m = \dim M$, $n = \dim N$, and $l > m \cdot n$. Let λ, μ be two partitions of l, and consider the composition:

$$M^{\otimes l} \xrightarrow{d_\lambda} M^{\otimes l} \xrightarrow{f^{\otimes l}} N^{\otimes l} \xrightarrow{d_\mu} N^{\otimes l}.$$

where $d_\lambda = \Gamma_{e_\lambda}$ is the graph of the idempotent e_λ on $(X \times Y)^l$. Recall further that the projectors d_λ commute with morphisms, so the map above is equal to $f^{\otimes l} \circ d_\lambda \circ d_\mu$. Since $e_\lambda \cdot e_\mu = 0$ if $\lambda \neq \mu$, we see that we get

$$f^{\otimes l} d_\lambda d_\mu = \begin{cases} 0 & \lambda \neq \mu \\ f^{\otimes l} d_\lambda & \text{else.} \end{cases} \quad (\text{as } d_\lambda \text{ are idempotents}).$$

Hence its enough to show the above composition vanishes for $l = nm+1$ and $\lambda = \mu$. Now suppose (for example) that $\Lambda^{m+1} M = 0$ any $\mathrm{Sym}^{n+1} N = 0$. Then the vanishing lemma proves the claim. ▨

Corollary: Suppose $M = (X, p, m)$ is both evenly and oddly f.d.. Then $M = 0$.

Proof: Apply the above to p. ▨

Now we can accomplish our goal.

Proposition: Let $M = (X, p, m) \cong M_+ \oplus M_-$ be a f.d. motive. If $M \cong M'_+ \oplus M'_-$ of even and odd f.d. motives, then $M'_+ \cong M_+$ and $M'_- \cong M_-$.

Proof: Suppose we had two decompositions. Write $p = p_+ + p_- = p'_+ + p'_-$. Then $p - p'_+$ is a projector which maps to M'_-, and hence $(p - p'_+) \circ p_+$ is smash-nilpotent, but goes from X to X, so must be nilpotent. Noting that $p = id_M$:

$$((p - p'_+) \circ p_+)^n = (p_+ - p'_+ \circ p_+)^n = 0 \Rightarrow p_+ = p_+^n = \underbrace{f(p_+, p'_+)}_{\text{Some expression.}} p'_+ \circ p_+$$

Since $p'_+ \circ p_+ : M_+ \to M'_+$, we must have $f: M'_+ \to M_+$, i.e., a section. But this implies that $M'_+ = M_+ \oplus K$, and so $\dim M'_+ \geq \dim M_+$. Proving the opposite equality shows $\dim K = 0$, but this implies that $K = 0$, so we're done ▨

Surjections in $\mathrm{Mot}_\sim(k)$

Let us work with chow motives for now.

Def: Let $f: M \to N$ be a morphism of motives. Then f is surjective if for all smooth projective varieties Z the induced map $CH(M \otimes ch(Z))_{\mathbb{Q}} \to CH(N \otimes ch(Z))_{\mathbb{Q}}$ is surjective.

Let me remind you that for $M = (X, p, m)$, $CH^i(M) = \mathrm{Im}(p_*: CH^{i+m}(X)_{\mathbb{Q}} \to CH^{i+m}(X)_{\mathbb{Q}}) \cong \mathrm{Hom}_{Mot}(\mathbb{L}^{\otimes i}, M)$, $\mathbb{L} = (\mathrm{Spec}\, k, id, -1)$.

Example: Let $\phi: X \to Y$ be a generically finite morphism of degree r. Then on motives, we have morphisms ϕ_* & ϕ^* s.t. $\phi_* \circ \phi^* = r\, id$. $\Rightarrow$ surjective.

Example: Consider the inverse of a blow-up $X \overset{\phi}{\dashrightarrow} Y = Bl_p X$ of a sm. proj. X at a point. Then $CH^1(Y) = CH^1(X) \oplus \mathbb{Z}[E]$, and $E \notin \mathrm{Im}\, \phi_*$ $\Rightarrow$ not surjective. in general dominant morphisms are surjective, but not dominant rational maps.

Lemma: Let $f: (X, p, m) \to (Y, q, n)$ be a morphism. Then TFAE:

i) f is surjective,
ii) $\exists$ a right inverse to f,
iii) $q = f \circ s$ for some $s \in Corr^0(Y, X)$.

Theorem: Let $f: M \to N$ be a surjective morphism of motives. If M is f.d., so is N.

Proof:

Step I: Suppose M is evenly (oddly) f.d. then the above lemma guarantees us a right inverse, $g: N \to M$, such that $f \circ g = id_N$. This induces a decomposition $M = N \oplus K \Rightarrow N$ and K are evenly (oddly) f.d.

Step II: Write $M = M_+ \oplus M_-$. One needs to show existence of $N = N_+ \oplus N_-$ such that $M_+ \twoheadrightarrow N_+$ and $M_- \twoheadrightarrow N_-$. Since the degree doesn't matter in the definition, we may take degrees zero, and regard f as a correspondence. Using the above lemma and M's decomposition, we get two endomorphisms $q'_\pm : N \to N$ $\big(M = (X, p, 0),\ N = (Y, q, 0)\big)$.

Step III: Show that there is a polynomial $P(t)$ such that $P(q'_\pm)$ are (almost) projectors. We set $q_\pm = q'^k_\pm \circ r_\pm$, $r_+ = P^k(q'_+)$, $r_- = P(q'^k_-)$.

Step IV: Show $M_\pm$ surjects onto $(Y, q_\pm, 0)$. ∎

This implies the following:

Corollary:

1) If $f: X \to Y$ is a dominant morphism with $ch(X)$ f.d., then $ch(Y)$ is also.
2) $M \oplus N$ f.d. $\Rightarrow$ M and N are f.d.
3) A motive which is dominated by a morphism from a finite product of curves is f.d. In particular the motive of an abelian variety is f.d.
4) Every summand of a tensor product of curves is f.d. They form a full tensor subcategory.

Now we have the following theorem:

Thm: Let $M=(X,p,m)$ and $f: M \to M$ be a morphism of Chow motives. Assume M is evenly (oddly) finite dimensional. We have:

1) There exists a nonzero polynomial $g(t) \in \mathbb{Q}[t]$ of degree $n-1$ with $g(f)=0$.
2) If $f \sim_{num} 0$, then $g(t) = t^{n-1}$.

Proof omitted.

Applications and Conjectures

We have functors $Mot_{rat} \longrightarrow Mot_{hom} \longrightarrow Mot_{num}$ taking $(X,p,m) \to (X, p_{hom}, m) \to (X, p_{num}, m)$. The first functor is not fully faithful, as if $Z \in Z^i(X)$ is a cycle, then Z is given by a morphism $f: \mathbb{L}^{\otimes i} \to M$. If $Z \not\sim_{rat} 0$ but $Z \sim_{hom} 0$, then this is an example. Explicitly on an elliptic curve E, $Z = a-b$ is not rationally zero, but is algebraically (hence homologically) zero.

Def: Let M be a chow motive and M_{hom} be it's image in Mot_{hom}. If $M_{hom} = 0$ yet $M \neq 0$, we say M is a phantom motive

Thm: If M is a f.d. chow motive, it is not a phantom motive.

Proof: Suppose it was. Then $p_{hom} \sim 0$, which implies $p_{num} \sim 0$. Since $M = M_+ \oplus M_-$, $p = p_+ + p_-$ are each numerically trivial, so $p_\pm^{n-1} = 0 \Rightarrow p_\pm = 0$, hence $M = 0$. ∎

Recall that if $H_+(M) = 0$ then $H(\mathrm{Sym}^n M) = \Lambda^n H_-(M)$, hence $H(\mathrm{Sym}^n M) = 0$ if $n > \dim H(M)$, and if $H_-(M) = 0$ then $H(\Lambda^n M) = \Lambda^n H_+(M)$, hence $H(\Lambda^n M) = 0$ if $n > \dim H(M)$.

Corollary: If M is a f.d. motive, $\dim M = \dim H(M)$.

Proof: It's enough to see it for M evenly f.d., so set $n = \dim M$ and note that since $\Lambda^n M \neq 0$, $H(\Lambda^n M) \neq 0$. Indeed motivic cohomology is defined via the image of the projector, hence nonzero. The above remarks show $\dim M \geq \dim H(M) = \dim H_+(M)$, and $H_+(\Lambda^n M) = \Lambda^n H_+(M)$, so $\dim H_+(M) \geq \dim M$. This shows equality. ∎

Conjecture (Kimura, O'Sullivan): Every Chow motive is f.d.

Conjecture $N(M)$, $N'(M)$: The ideal $J_M = \mathrm{Ker}(\mathrm{End}_{rat}(M) \longrightarrow \mathrm{End}_{num}(M))$ is nilpotent, resp a nil-ideal (every element is nilpotent).

Thm: Kimura + O'Sullivan $\Rightarrow$ $N(M)$ + no phantom motives

Proof

It is a non commutative algebra fact that if a nil-ideal has the order of nilpotency unif. bounded, then it is nilpotent. Now let f be an endomorphism of a f.d. motive which is numerically trivial. We aim to show it's nilpotent.

We have $f = (p_+ + p_-) \circ f \circ (p_+ + p_-) = f_+ + f_- + f_{mix}$, f_{mix} does not preserve parity $\Rightarrow f^r \sim_\otimes 0$, r independent of f. Then expanding f^n, we get as an average monomial:

$$f_\pm^{k_1} \circ f_{mix}^{l_1} \circ \cdots \circ f_{mix}^{l_i} \circ f_\pm^{k_i} \qquad i \leq l_1 + \cdots + l_r \leq r-1.$$

Since f (hence $f_\pm$) are numerically trivial gives $f_\pm^s = 0$. Then $f^n = 0$ for $n \geq rs$.

Now recall that Voevodsky's conjecture asks whether a numerically trivial $f: M \to M$ is smash nilpotent (note this implies actual nilpotence). We claim this implies Kimura-O'Sullivan. Indeed Voevodsky implies $D(X)$ ($\sim_{hom} = \sim_{num}$), but $D(X) \Rightarrow A(X,L)$. If this holds for all X, we get $B(X) \Rightarrow C(X)$. But Voevodsky (by def) implies $N'(X)$, and $C(X)$ implies $S(X)$ (the sign conjecture we skipped). It is now a theorem of Jannsen that this gives $ch(X)$ f.d. for all X.